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Analysis of large deflection equilibrium states of composite shells of revolution. Part 1. General model and singular perturbation analysis

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Abstract

The singular perturbation method is applied in combination with the variational method to the general Reissner's equations describing axially symmetric large deflections of thin composite shells of revolution with varying material and geometrical parameters in meridian direction. The obtained asymptotic nonlinear boundary value problem is significantly simpler in comparison with the original one. The asymptotic model has the following advantages: number of the geometrical and stiffness parameters of shell is effectively reduced, and singularities are eliminated without loss of the accuracy of the solution. The simple asymptotic formulae have been derived in case of completed shells. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Study of equilibrium states of shells with large deflections is of a practical interest for two reasons. First, the information concerning the initial post-buckling behavior is frequently not sufficient for analysis of stability of thin shells due to the great imperfection sensitivity of such systems. It means that small imperfections both in geometry and loading can result in strong reduction of the classical critical load obtained for perfect shell. Koiter's perturbation method (Koiter, 1969) and other standard methods of bifurcation theory fundamentally furnish the possibility of taking into account only small imperfections. This results in a critical load value which is close to a classical one. However, the ranges of admissible loads of real structures are restricted by significantly smaller values and conform to nonclose and nonimminent post-buckling equilibrium states of shells which are characterized by large displacements. Thus the post-buckling path in the load–displacement diagram with large deflections could provide additional required

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information on post-buckling behavior of shells in order to be able to estimate stability. Moreover, the methods of local or standard bifurcation theory based on using the eigenfunctions of the appropriate linear buckling problem, is not applicable to many practically important problems for shells, such as circular cylinders under axial compression or spheres under external pressure, because of a large number of closely spaced eigenvalues. Furthermore, experiments and numerical and analytical solutions (Berke and Carlson, 1968; Evkin and Dubichev, 1997a,b; Pogorelov, 1967; Babenko, 1977; Kriegsmann and Lange, 1980; Lange and Kriegsmann, 1981; Graff et al., 1985) show that shell equilibrium configurations with large deflections are quite different from classical buckling modes. They consist of regions of inextensional strain (for example, inverted part of sphere) surrounded by narrow zones of extensive bending and membrane deformations (inner boundary layers). Investigation of such configurations calls for development of new sophisticated methods of analysis.

A second reason is that structures with thin-walled elements whose application involves large displacements, rotations and deformations are recently used intensively, for example, metallic bladders for aircraft cryogenic fluid tanks and propulsion systems. The typical problem in this case is to design a structure with a given load–displacement diagram and to estimate stress and strain states of appropriate equilibrium shapes.

This paper continues the series of investigations (Pogorelov, 1967; Babenko, 1977; Ranjan and Steele, 1977; Wan, 1978; Kriegsmann and Lange, 1980; Lange and Kriegsmann, 1981; Graff et al., 1985; Evkin, 1989, 1995; Evkin and Korovaitsev, 1992; Korovaitsev and Evkin, 1992; Evkin and Dubichev, 1995, 1997a), in which the asymptotic and geometrical methods were elaborated to obtain the solution of nonlinear shell equations. The main purpose of this study is, on the first hand, to develop the most general approach to this problem; and on the other hand, to obtain the simple analytical solution with accuracy similar to the one of the original thin shell model. The generality of the approach is based on a choice of the initial mechanical model. We consider anisotropic composite shell of revolution with geometrical and stiffness parameters, that vary in the meridional direction, and we assume the structure and its deformations to be axisymmetric. In contrast to works of other authors (Babenko, 1977; Kriegsmann and Lange, 1980; Lange and Kriegsmann, 1981; Graff et al., 1985) in which the singular perturbation analysis was applied to the spherical shell under external pressure, in the present paper we combine the asymptotic and variational methods. This allows us to consider only the leading order asymptotic boundary value problem and, in all cases, to reduce the initial complex nonlinear problem with a singularity and a large number of geometrical and stiffness parameters to a much simpler one without singularities and with a maximum of one parameter. The latter one is equal to zero for the shell which is symmetrical with respect to its midsurface. We have obtained the numerical solution of a respectable problem and derived simple resultant asymptotic formulae describing load–displacement diagrams and stress states of shells of revolution with large deflections. The error of obtained results is the same one, implicit in the initial Reissner's equations, because these equations can be obtained from the three-dimensional theory of elasticity using the asymptotic method with the same small parameter as we use in the suggested approach.

2. Mechanical model and shell equations

Our study is based on the Reissner's equations (Reissner, 1969, 1972) in their most general form. This model contains the following two main restrictions. The first one concerns thin shell theory based on Kirchhoff's hypotheses. This restraint is of a principal character in our asymptotic analysis, because of an assumption of smallness of ratio of shell thickness to radius of curvature.

The second restriction is related to the axial symmetry of a structure and its deformation. The effective methods capable of dealing with two-dimensional nonlinear shell theory problems are not available yet, because of rather complicated mathematical problems due to the high multiplicity of the critical eigenvalue and singularities. On the other side, the axisymmetric structures are very important from a practical point of view. A very important problem in designing transformable thin-walled structural elements is to ensure axial symmetry of their deformation with large displacements, because of the intended technological functions of structures and because of lack of structural materials capable of withstanding strong bending in two directions. This can be achieved by using orthotropic composite structures with improved flexural rigidity in the circumferential direction. For example, in a paper (Evkin and Dubichev, 1995) the conditions of bifurcation of axisymmetrical equilibrium configuration to nonsymmetric one have been derived for large bending deflections of orthotropic shallow sphere. Besides, the experimental investigations (Berke and Carlson, 1968; Evkin and Dubichev, 1997b) in spherical shell stability field point out the special importance of studying the symmetrical shapes. In the present study, we consider a composite shell with geometrical and stiffness parameters variable in both meridian and transversal directions.

Reissner's equations describe arbitrarily large deflections, rotations and deformations of a thin shell of revolution. According to them, the basic relations that link displacements and deformations of points on the midsurface of the shell can be presented as follows:

$$\varepsilon_2 = \frac{u}{r_0}, \qquad \frac{du}{ds} = (1+\varepsilon_1)\cos\psi - \cos\psi_0 \tag{2.1}$$

$$\frac{dv}{ds} = (1+\varepsilon_1)\sin\psi - \sin\psi_0 \tag{2.2}$$

$$\chi_1 = \frac{d\psi}{ds} - \frac{1+\varepsilon_1}{R_1}, \qquad \chi_2 = \left(\frac{\sin\psi}{r} - \frac{\sin\psi_0}{r_0}\right)(1+\varepsilon_1) \tag{2.3}$$

where ε_1 and ε_2 are the deformations of midsurface element in the meridional and circumferential directions, u and v are respectively the radial (horizontal) and axial (vertical) displacement components, ψ_0 and ψ are the angles of inclination of the tangent to the meridian before and after shell deformation (Fig. 1),

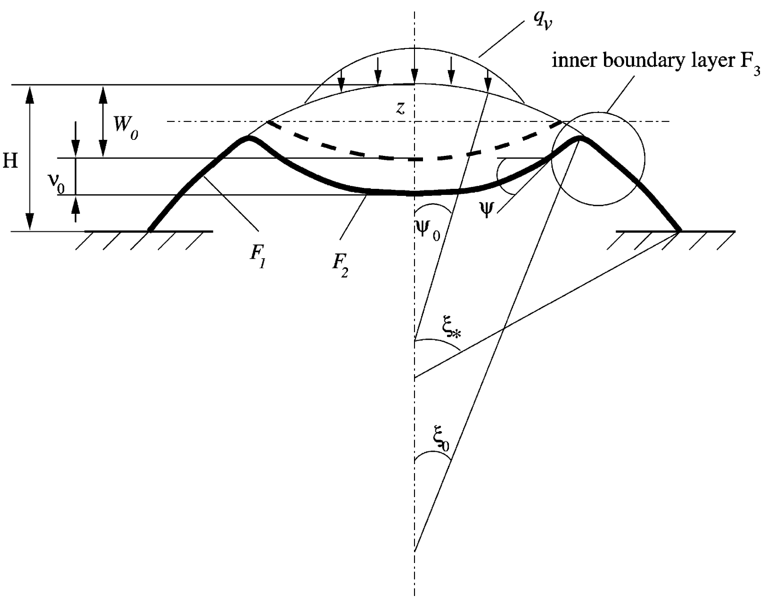

Fig. 1. Axisymmetrical equilibrium configuration of thin shell of revolution with large deflections.

$1/R_1$, $1/R_2$ and χ_1, χ_2 are the curvatures and their changes in meridional and circumferential directions respectively, s is the independent variable, which means the distance from the coordinate origin along an arc of the meridian to the point being considered; whereas, r_0 and r are the distances from this point to the symmetry axis z, subscript 0 indicates properties of underformed shell, 1 is meridional, and 2 is circumferential. We have also evident formulae

$$r_0 = R_2 \sin\psi_0, \quad r = r_0 + u, \quad \frac{1}{R_1} = \frac{d\psi_0}{ds} \tag{2.4}$$

To study the equilibrium configurations we load the shell by an external pressure q quasistatically. The differential equilibrium equations of a shell element have the form

$$\frac{dV}{ds} = -(1+\varepsilon_1)\left(\frac{\cos\psi}{r}V + q_v\right) \tag{2.5}$$

$$\frac{dU}{ds} = -(1+\varepsilon_1)\left(\frac{\cos\psi}{r}U - \frac{N_2}{r} + q_u\right) \tag{2.6}$$

$$\frac{dM_1}{ds} = -(1+\varepsilon_1)\left[\frac{\cos\psi}{r}(M_1 - M_2) - U\sin\psi + V\cos\psi\right] \tag{2.7}$$

Here q_u and q_v, U and V are the horizontal and vertical components of the external load q and stress resultant of deformed shell respectively, N_1 and M_1 are the stress resultant and bending moment that arise in a unit area perpendicular to the meridian, N_2 and M_2 are the corresponding internal force and moment loads that act on a unit area in the plane of the meridian. All stress resultants are defined as being per unit deformed length of the shell. There is obvious relation between introduced forces

$$N_1 = U\cos\psi + V\sin\psi \tag{2.8}$$

We use formulae linking generalized internal loads and axially symmetric deformations in the general form

$$N_i = B_{ij}\varepsilon_j + K_{ij}\chi_j, \quad M_i = D_{ij}\chi_j + K_{ij}\varepsilon_j \qquad (i,j = 1,2) \tag{2.9}$$

that allows us to consider shells without symmetry with respect to their midsurfaces (for instance, laminated composite structures (Kalamkarov, 1992)).

To complete the formulation of the initial boundary value problem we introduce the classical types of boundary conditions (at $\psi_0 = \xi_*$) in following forms:

$$\begin{aligned} &\psi = \xi_*, \quad u = 0, \quad v = 0 \\ &\psi = \xi_*, \quad U = 0, \quad v = 0 \\ &M_1 = 0, \quad u = 0, \quad v = 0 \end{aligned} \tag{2.10}$$

The first condition corresponds to the shell clamped along meridian $\psi_0 = \xi_*$, the second condition corresponds to an edge resting on a frictionless surface, and the last one corresponds to the simply supported edge. We confine ourselves to consideration of shells with simply connected domain $0 \leqslant \psi_0 \leqslant \xi_*$. Forming the solution, we will take into account the symmetry conditions at the shell pole (at $\psi_0 = 0$).

3. Singular perturbation analysis

3.1. Asymptotic analysis of boundary value problem

The above formulated boundary value problem is rather complicated due to the presence of a large number of structural properties (nine stiffness and at least two geometrical parameters) changing in the meridional direction. To simplify the initial problem and to reveal the contained implicit singularity we apply the asymptotic method introducing the nondimensional parameter in rather general form

$$\varepsilon^2 = \frac{R_2 h}{R_1^2}\sqrt{\frac{D}{a_* B_{22} h^2}} \tag{3.1}$$

where

$$D = D_{11} - \frac{B_{22}K_{11}^2 - 2B_{12}K_{12}K_{11} + B_{11}K_{12}^2}{B_{11}B_{22} - B_{12}^2}, \qquad a_* = 1 - \frac{B_{12}^2}{B_{11}B_{22}}.$$

To understand the meaning of this parameter let us consider several particular cases. For the orthotropic shell of revolution, which is symmetrical with respect to its middle surface (without eccentricity), we obtain the following expression:

$$\varepsilon^2 = \frac{R_2 h}{R_1^2}\sqrt{\frac{D_{11}}{a_* B_{22} h^2}} \tag{3.2}$$

For the case of isotropic shell we have

$$\varepsilon^2 = \frac{hR_2}{R_1^2\sqrt{12a_*}} \tag{3.3}$$

Finally, for the isotropic sphere we derive the well-known parameter

$$\varepsilon^2 = \frac{h}{R\sqrt{12a_*}} \tag{3.4}$$

which is small for a thin structure and was successfully used in the asymptotic analysis of shell theory boundary value problems. Returning step by step to the initial definition (3.1) of parameter ε^2 we conclude that it remains small for all thin shells of revolution except those, which are strongly improved by flexural rigidity in meridional direction. This condition has the following mathematical expression:

$$\frac{D_{11}}{a_* B_{22} h^2} \sim \mathrm{O}(1) \tag{3.5}$$

Besides, a class of shells of revolution for which the radii of the principal curvatures are the same order of magnitude is considered:

$$\frac{R_1}{R_2} \sim 1 \tag{3.6}$$

In addition, it is assumed that the shape of the shell and its rigidity properties vary smoothly in the meridional direction. Then

$$\frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}\psi_0} \sim 1 \tag{3.7}$$

where F is an arbitrary characteristic (including external pressure) of the undeformed system. The above restrictions (3.5)–(3.7) are not strong and have an asymptotic meaning for $\varepsilon^2 \to 0$. They allow us to obtain analytical solutions for a rather large class of thin shells of revolution.

To facilitate the asymptotic analysis we introduce new basic dimensionless variables, which are of the order of magnitude of unity and are marked with over-bars in the following formulae:

$$N_2 = \varepsilon^{k_1-1} q_* R_2 \sin\psi_0 \overline{N}_2, \quad U = 0.25\varepsilon^{k_2} R_1 q_* \overline{U}, \quad V = \varepsilon^{k_3+1} q_* R_1 \overline{V} \tag{3.8}$$

$$\varepsilon_{12} = \varepsilon^{k_4+1}\bar{\varepsilon}_1, \qquad \varepsilon_2 = \varepsilon^{k_4+1}\bar{\varepsilon}_2 \tag{3.9}$$

$$\chi_1 = \varepsilon^{k_5-1}\overline{\chi}_1/R_1, \qquad \chi_2 = \varepsilon^{k_6}\overline{\chi}_2/R_2 \tag{3.10}$$

$$\psi = \varepsilon^{k_7}\overline{\psi} \tag{3.11}$$

$$q_v = \varepsilon^{k_8} q_* \bar{q}_v, \quad q_u = \varepsilon^{k_8} q_* \bar{q}_u, \quad q_* = \frac{4}{R_1 R_2}\sqrt{a_* D B_{22}} \tag{3.12}$$

It should be noted that in the special case of an isotropic spherical shell the value of external pressure q_* coincides with the classical critical one.

Relations (3.8)–(3.12) represent only the leading terms of asymptotic expansions. All new variables are multiplied by the parameter ε of different orders, which should be chosen during asymptotic analysis. As far as it is hard to describe the routine process of obtaining appropriate exponents k_i among integers, it is not difficult to verify the final result. We have obtained that $k_i = 0$ as $i = 1, 2, \ldots, 7$ and $k_8 = 0$ or 1. Naturally, we cannot guarantee the uniqueness of derived solution because of nonlinearity of boundary value problem, but we intend to show the importance and give a mechanical interpretation of obtained results.

We are now in a position to simplify the given Reissner's equations according to asymptotic relations (3.8)–(3.12) with $k_i = 0$. From (2.3) and (2.8) we derive that

$$\overline{\chi}_1 = \varepsilon\frac{\mathrm{d}\psi}{\mathrm{d}\psi_0} \tag{3.13}$$

and

$$N_1 = 0.25 R_1 q_* \overline{U}\cos\psi \tag{3.14}$$

The last relation and formulae (3.8)–(3.11) lead to the asymptotic model, in which bending in the meridional direction ($\chi_1 \gg \chi_2, 1/R_1, 1/R_2$) and circumferential stress ($N_2 \gg N_1, U, V$) are dominant. Note that this kind of deformation is commonly considered in the linear theory of boundary effects of thin shells. However, as we will see, a possibility of existence of *inner* boundary layers emerges in thin shells with large deflections due to the nonlinearity of system. Besides, it follows from Eq. (3.9) that deformations are small compared to unity. From Eq. (3.11) we also see that the angle of rotation ψ is not small. In opposite case, we have to modify the definition of small parameter according to Evkin (1989) and consider the ratio of shell thickness to the deflection amplitude instead of parameter given by Eq. (3.1). But the final results will be the same in both cases.

The stress–deformation relations become

$$\varepsilon_1 = -\frac{1}{B_{11}}(B_{12}\varepsilon_2 + K_{11}\chi_1), \qquad N_2 = \varepsilon_2 B_{22} a_* + \chi_1\left(K_{21} - \frac{B_{21}K_{11}}{B_{11}}\right) \tag{3.15}$$

$$M_1 = \chi_1\left(D_{11} - \frac{K_{11}^2}{B_{11}}\right) + \varepsilon_2\left(K_{12} - \frac{K_{11}B_{12}}{B_{11}}\right) \tag{3.16}$$

$$M_2 = \chi_1\left(D_{21} - \frac{K_{21}K_{11}}{B_{11}}\right) + \varepsilon_2\left(K_{22} - \frac{K_{21}B_{12}}{B_{11}}\right) \tag{3.17}$$

from which we derive formulae expressed in terms of main characteristics N_2 and χ_1

$$\varepsilon_2 = \frac{N_2}{a_* B_{22}} - \chi_1 C, \qquad M_1 = \chi_1 D + N_2 C, \quad C = \frac{K_{12}B_{11} - K_{11}B_{12}}{B_{11}B_{22} - B_{12}^2} \tag{3.18}$$

In case of isotropic shells without eccentricity we arrive to

$$M_1 = D_{11}\chi_1, \quad M_2 = \frac{D_{21}}{D_{11}}M_1, \quad N_2 = a_* B_{22}\varepsilon_2, \quad \varepsilon_1 = -\frac{B_{12}}{B_{11}}\varepsilon_2 \tag{3.19}$$

Let us simplify the equilibrium equations. From Eq. (2.5) we have

$$V = \frac{1}{R_2 \sin\psi_0}\int_0^{\psi_0} R_1 R_2 \sin\psi_0 q_* \bar{q}_v \mathrm{d}\psi_0 \tag{3.20}$$

Eq. (2.6) becomes

$$0.25\varepsilon\frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}\psi_0}\left(q_* R_1 \overline{U}\right) = q_* \overline{N}_2 \tag{3.21}$$

Taking into account Eqs. (3.20), (3.21), (3.18) and (3.13), we obtain the last equilibrium equation (2.7) in the final form

$$\frac{1}{R_1^2 q_*}\frac{\mathrm{d}}{\mathrm{d}\psi_0}\varepsilon^2 q_* R_1^2 \frac{\mathrm{d}\psi}{\mathrm{d}\psi_0} + \varepsilon^2 n \sin\psi_0 R_1 \frac{\mathrm{d}}{\mathrm{d}\psi_0}\left(R_1 q_* \overline{U}\right) = \overline{U}\sin\psi + \hat{q}_v\cos\psi \tag{3.22}$$

where

$$\hat{q}_v = \frac{4}{q_* R_1 R_2 \sin\psi_0}\int_0^{\psi_0} R_1 R_2 \sin\psi_0 q_* \bar{q}_v \mathrm{d}\psi_0, \quad n = C\sqrt{\frac{a_* B_{22}}{D}} \tag{3.23}$$

Joining two relations (2.1) we have equation

$$\frac{\mathrm{d}(\varepsilon_2 r_0)}{\mathrm{d}s} = (1+\varepsilon_1)\cos\psi - \cos\psi_0 \tag{3.24}$$

which can be represented using Eqs. (3.18), (3.21) and (3.13) in the conclusive form

$$\frac{1}{R_1}\frac{\mathrm{d}}{\mathrm{d}\psi_0}\varepsilon^2\frac{\sin^2\psi_0}{q_*}\frac{\mathrm{d}}{\mathrm{d}\psi_0}\left(R_1 q_* \overline{U}\right) - \varepsilon^2 n \sin\psi_0 R_1 \frac{\mathrm{d}\psi}{\mathrm{d}\psi_0} = \cos\psi - \cos\psi_0 \tag{3.25}$$

We have obtained the governing system of two equations (3.22) and (3.25), which contains two variables $\overline{U}$ and ψ depending on ψ_0, only one stiffness characteristic n and small parameter ε^2. We have derived simplification of the initial Reissner's equations due to asymptotic analysis, which leads to the clear physical model of a thin shell with the large deflections and not small angles of rotation ψ: curvature change in the meridional direction is much larger than initial curvatures of underformed shell and its curvature change in the circumferential direction ($\chi_1 \gg \chi_2, 1/R_1, 1/R_2$); circumferential stress resultant is much larger than other membrane stress resultants ($N_2 \gg N_1, U, V$); strains are small compared with unit: ε_1, $\varepsilon_2 \ll 1$. It should be noted here, that we have used all the Reissner's equations except Eq. (2.2), checking their consistency according to the asymptotic representations (3.8)–(3.12). Eq. (2.2) serves for evaluation of the vertical displacement v after deriving the solution of main problem (3.22) and (3.25) as well as Eq. (3.20) serves for calculation of the vertical reaction on the shell edge.

The main feature of the governing equations is the presence of the small parameter ε^2 at the higher-order derivatives. The left-hand sides of both Eqs. (3.22) and (3.25) vanish upon setting $\varepsilon^2 = 0$, and the reduced system has two obvious solutions. The first one

$$\psi = \psi_0, \qquad \overline{U} = -\hat{q}_v \cot\psi_0 \tag{3.26}$$

corresponds to the initial trivial state of shell (region F_1 on Fig. 1). The second one

$$\psi = -\psi_0, \qquad \overline{U} = \hat{q}_v \cot\psi_0 \tag{3.27}$$

is a reflection of part F_2 of initial midsurface of shell with respect to some plate, which is normal to the axis of symmetry z. Moreover, the latter solution allows also the vertical translation of part F_2 as a rigid body. Although it is of a significant importance, the possibility of translation of the shell's part was omitted in previous investigations. Actually, we can construct several combinations (cascade) of such types of solutions in some regions, but, in practical sense, the most important one is sketched on the Fig. 1. It is important, that both solutions (3.26) and (3.27) satisfy the reduced system as well as the initial full Reissner's equations.

We should emphasize the mechanical meaning of inverted shape of the shell. Namely, it can be attained by isometric transformation of the midsurface of a shell without extensional strain. It is clear that such configurations are more preferable for thin-walled structures with a flexural rigidity significantly smaller than the extensional one. Such forms, as well as narrow regions of extensive deformations connecting smooth parts of shape were observed in a number of experiments. These regions correspond to the inner boundary layers joining together in a smooth manner two solutions of type (3.26) and (3.27) obtained in first stage of singular perturbation analysis. The second stage is to construct fast changing boundary layer functions, which we introduce in the form

$$\overline{U} = \overline{U}(\xi), \qquad \psi = \psi(\xi), \tag{3.28}$$

where $\xi = (\psi_0 - \xi_0)/\varepsilon$, ξ_0 is an unknown angle defining the location of the boundary layer of width $O(\varepsilon)$. The presence of inner boundary layers, whose position on the shell surface is unknown in advance, significantly complicates the solution of fully nonlinear shell equations.

For arbitrary, but not too small ξ_0, we obtain the following equations

$$\frac{d^2\psi}{d\xi^2} + \sin\xi_0 n \frac{d^2\overline{U}}{d\xi^2} = \overline{U}\sin\psi + \hat{q}_v\cos\psi \tag{3.29}$$

$$\frac{d^2\overline{U}}{d\xi^2} - \frac{n}{\sin\xi_0}\frac{d^2\psi}{d\xi^2} = \frac{\cos\psi - \cos\xi_0}{\sin^2\xi_0} \tag{3.30}$$

where

$$n = n(\xi_0), \hat{q}_v = \hat{q}_v(\xi_0) = \frac{4}{q_* R_1 R_2 \sin\xi_0}\int_0^{\xi_0} R_1 R_2 \sin\psi_0 q_* \bar{q}_v \, d\psi_0.$$

In order to complement the specification of problem we introduce the parameter defining the location of the shell edge

$$\xi_1 = (\xi_* - \xi_0)/\varepsilon \tag{3.31}$$

and taking into account Eqs. (2.1), (3.18), (3.21) and (3.13) we rewrite the boundary conditions (2.10) in following form (at $\xi = \xi_1$)

$$\psi = \xi_*, \qquad \overline{U}' - \frac{n}{\sin \xi_*}\psi' = 0 \tag{3.32}$$

$$\psi = \xi_*, \qquad \overline{U} = 0 \tag{3.33}$$

$$\psi' + n \sin \xi_* \overline{U}' = 0, \qquad \overline{U}' - \frac{n}{\sin \xi_*}\psi' = 0 \tag{3.34}$$

Besides, we impose conditions of asymptotic transfer of boundary layer functions to solution (3.27) (if ξ_0 is not too small, which means that boundary layer region is located sufficiently far from the shell pole)

$$\psi = -\xi_0, \qquad \overline{U} = \overline{U}_0 \quad \text{as } \xi \to -\infty \tag{3.35}$$

where $\overline{U}_0 = \hat{q}_v(\xi_0) \cot \xi_0$.

Concerning the position of boundary layer we will distinguish two cases. The first one corresponds to the boundary layer location close to the edge of shell, if $(\xi_* - \xi_0) \sim \varepsilon$. Then we can replace ξ_* by ξ_0 in formulae (3.32)–(3.34). On the other hand, when the boundary layer is sufficiently far from the shell border, we have to satisfy the transfer conditions of boundary layer functions to the trivial solution (3.26):

$$\psi = \xi_0, \qquad \overline{U} = -\overline{U}_0 \quad \text{as } \xi \to +\infty \tag{3.36}$$

Solving the main boundary value problem we can link the deflection amplitude with the load parameter. Then using the solution of this problem $\overline{U}(\xi)$, $\psi(\xi)$ and Eqs. (3.13), (3.10), (3.21), (3.8), (3.18), (3.17) we can return to the initial variables and obtain asymptotic estimations of stress state of shells according to the following simple and explicit formulae:

$$N_1 = 0, \qquad N_2 = \frac{D^{1/4}(a_* B_{22})^{3/4}}{\sqrt{R_2}} \sin \xi_0 \overline{U}'(\xi) \tag{3.37}$$

$$M_1 = \frac{D}{\varepsilon R_1}(\psi'(\xi) + n \sin \xi_0 \overline{U}'(\xi)) \tag{3.38}$$

$$M_2 = \frac{D}{\varepsilon R_1}(D^* \psi'(\xi) + n^* \sin \xi_0 \overline{U}'(\xi)) \tag{3.39}$$

where

$$D^* = \frac{D_{21}}{D} - \frac{K_{21}K_{11}}{B_{11}D} - \frac{BC}{D}, \quad B = \frac{B_{11}K_{22} - K_{21}B_{21}}{B_{11}}, \quad n^* = B\sqrt{a_* B_{22} D} \tag{3.40}$$

In the paper (Kriegsmann and Lange, 1980) the authors have obtained, that $q_0 \to 0$ for completed spherical shell or by receding of the inner boundary layer far enough from the shell boundary, and that load parameter was not contained in leading asymptotic equations (3.29) and (3.30). Therefore, solving them only, it is not possible to establish the relation of the load versus the parameters of shell equilibrium configurations. To handle this problem it was suggested to take into consideration the next asymptotic approximation (Kriegsmann and Lange, 1980; Graff et al., 1985). But it is clear that by locating the boundary layer far enough from the shell edge the problem can be simplified because of its independence from types of edge support. In order to analyze this and to gain deeper understanding of the asymptotic approach we will invoke the variational principle.

3.2. Variational approach to the asymptotic analysis of the problem

We use the Lagrange principle, according to which the functional of total potential energy $E = W - A$ of system must attain a stationary value that corresponds to equilibrium state of the shell. Here W is the potential energy of structure and A is the work of the constant external load on the admissible displacements. Evaluating W and A we distinguish two regions (Fig. 1): the first one F_3 corresponds to the boundary layer, the second one F_2 corresponds to the inverted segment described by the smooth part of asymptotic solution. Using leading asymptotic components of stress and strain components of the shell we obtain the formula for the boundary layer deformation energy

$$W_1 = 0.5 \int\int_{F_3} (M_1\chi_1 + N_2\varepsilon_2)\,\mathrm{d}F_3 \tag{3.41}$$

which can be represented according to above obtained relations in the simple form

$$W_1 = a_* \pi B_{22} R_1^3 \xi_0^2 \sin\xi_0 \varepsilon^3 J / R_2 \tag{3.42}$$

where

$$J = \int_{-\infty}^{\xi_1} \left[f^2 + (\varphi')^2 \right] \mathrm{d}\xi \tag{3.43}$$

and

$$f = \frac{\sin\xi_0}{\xi_0} \overline{U}', \quad \varphi = \psi/\xi_0. \tag{3.44}$$

To calculate W_1 we have to solve asymptotic equations (3.29) and (3.30) with related boundary conditions. It is not necessary for the evaluation of

$$W_2 = \int\int_{F_2} \frac{D_{11}}{2} \left(\chi_1^2 + \frac{D_{22}}{D_{11}} \chi_2^2 + \frac{2D_{12}}{D_{11}} \chi_1 \chi_2 \right) \mathrm{d}F_2 \tag{3.45}$$

where $\chi_1 = -2/R_1$, $\chi_2 = -2/R_2$ in the inverted inextensional part of shell. For example,

$$W_2 = 4\pi D_{22}(1 - \cos\xi_0) \tag{3.46}$$

for the spherical shell without eccentricity. Moreover, if

$$\frac{D_{22}}{D_{11}} \sim \mathrm{O}(1), \qquad \frac{D_{12}}{D_{11}} \sim \mathrm{O}(1) \tag{3.47}$$

then $W_2/W_1 \sim \varepsilon$ and we can drop W_2.

It is clear, that evaluating work A of external load we have to take into account only corresponding displacements in domain F_2 because they are asymptotically small in the boundary layer region. Thus

$$A = \int\int_{F_2} q_v v\,\mathrm{d}F_2 \tag{3.48}$$

We distinguish two types of displacements in domain F_2 ($0 \leqslant \psi_0 \leqslant \xi_0$), which are sketched in the Fig. 1. The first one (dashed line with amplitude of deflection W_0 in the Fig. 1) is a reflection of a part of the initial surface with respect to some plate, which is normal to the axis of symmetry. Work denoted as A_1, corresponds to the above displacement. The work denoted as A_2 corresponds to additional translation of part F_2 as a rigid body (with vertical displacement v_0 shown in the Fig. 1). We can obtain the total amount of work as sum $A = A_1 + A_2$. Let us derive the formula for calculation of $A_1(\xi_0)$ assuming that the initial midsurface of shell of revolution is given in the general parametric form

$$z = z(\psi_0), \qquad r_0 = r_0(\psi_0) \tag{3.49}$$

Then the vertical displacement function of first type of displacement can be represented by formula

$$v = 2[z(\xi_0) - z(\psi_0)], \quad 0 \leqslant \psi_0 \leqslant \xi_0 \tag{3.50}$$

We have also

$$A_1 = 4\pi \int_0^{\xi_0} [z(\xi_0) - z(\psi_0)] r_0 q_v R_1 \, \mathrm{d}\psi_0 \tag{3.51}$$

Using Eq. (3.51) we can obtain explicit expressions for work A_1 if shell surface and loading form are given, because A_1 does not depend on unknown boundary layer functions. For example, for uniform external pressure we derive

$$\frac{\partial A_1}{\partial \xi_0} = 2\pi q R_2^2 \sin^2 \xi_0 \frac{\mathrm{d}z(\xi_0)}{\mathrm{d}\xi_0} \tag{3.52}$$

For the vertical force Q concentrated in the shell pole we have $A_1 = QW_0$,

$$W_0 = 2[z(\xi_0) - z(0)] \quad \text{and} \quad \frac{\partial A_1}{\partial \xi_0} = 2Q \frac{\mathrm{d}z(\xi_0)}{\mathrm{d}\xi_0} \tag{3.53}$$

The expression for another summand A_2 of the total work is more complicated due to its dependence not only on global parameter of the equilibrium configuration ξ_0, but also on boundary layer functions, which are unknown in advance. We have

$$A_2 = v_0 \int_0^{\xi_1} 2\pi r_0 q_v d \tag{3.54}$$

According to Eq. (3.23)

$$A_2 = \pi \hat{q}_v q_* R_1 R_2 \sin \xi_0 v_0 \tag{3.55}$$

From Eq. (2.2) we derive

$$v_0 = \int_0^{\xi_1} (\sin \xi_0 - \sin \psi) \, \mathrm{d}\xi - \int_{-\infty}^{0} (\sin \xi_0 + \sin \psi) \, \mathrm{d}\xi \tag{3.56}$$

or

$$v_0 = -\varepsilon R_1 \int_{-\infty}^{\xi_1} \sin(\xi_0 \varphi) \, \mathrm{d}\xi + \Delta(\xi_0, \xi_1) \tag{3.57}$$

where the second summand does not depend on boundary layer functions. Dropping it for our further purposes, we obtain the final expression

$$A_2 = -a_* \pi B_{22} R_1^3 \xi_0^2 \sin \xi_0 \varepsilon^3 \frac{2\hat{q}_v}{\xi_0^2 R_2} \int_{-\infty}^{\xi_1} \sin(\xi_0 \varphi) \, \mathrm{d}\xi \tag{3.58}$$

We will consider two steps using the variational principle. At the first step we will take into account only those summands of energy, which depend on boundary layer functions $f(\xi)$ and $\varphi(\xi)$. At this step we can drop component A_1, which does not depend on $f(\xi)$ and $\varphi(\xi)$ and consider ξ_0 as a constant. After obtaining solution for boundary layer functions we will move to the second step at which we will consider energy varying in ξ_0. We have the total potential energy of the system at the first step in the following form:

$$E = W_1 - A_2 = a_* \pi B_{22} R_1^3 \xi_0^2 \sin \xi_0 \varepsilon^3 J_0 / R_2 \tag{3.59}$$

Here

$$J_0 = \int_{-\infty}^{\xi_1} \left[f^2 + (\varphi')^2 + \frac{2\hat{q}_v}{\xi_0^2} \sin(\xi_0 \varphi) \right] d\xi \tag{3.60}$$

is the functional, which variation in the functions $f(\xi)$ and $\varphi(\xi)$ we have to consider jointly with the Eq. (3.30) represented in the following form:

$$f' - n\varphi'' = \frac{\cos(\xi_0\varphi) - \cos\xi_0}{\xi_0 \sin\xi_0} \tag{3.61}$$

Using Lagrange's factor λ_1 we obtain the functional

$$J_0 = \int_{-\infty}^{\xi_1} \left[f^2 + (\varphi')^2 + \frac{2\hat{q}_v}{\xi_0^2} \sin(\xi_0\varphi) + \lambda_1 \left(f' - n\varphi'' - \frac{\cos(\xi_0\varphi) - \cos\xi_0}{\xi_0 \sin\xi_0} \right) \right] d\xi \tag{3.62}$$

variation of which in λ_1, f and φ leads to Eq. (3.61) or Eq. (3.30) and to equations

$$2f - \lambda_1' = 0 \tag{3.63}$$

$$\frac{\lambda_1 \sin(\xi_0\varphi)}{\sin\xi_0} - 2\varphi'' - n\lambda_1'' + \frac{2\hat{q}_v}{\xi_0} \cos(\xi_0\varphi) = 0 \tag{3.64}$$

Formula (3.63) together with Eq. (3.44) yields

$$\lambda_1 = \frac{2\sin\xi_0}{\xi_0} \overline{U} \tag{3.65}$$

and instead of Eq. (3.64) we arrive at the equilibrium equation (3.29).

In the case, when boundary layer region is close to the shell border ($\xi_* - \xi_0 \sim \varepsilon$) we do not obtain new useful information except a variational point of view on the problem and verification of asymptotic equations (3.29) and (3.30). But in case if the inner boundary layer is located sufficiently far from edge of shell ($(\xi_* - \xi_0)/\varepsilon \gg 1$) we arrive at the following asymptotic simplifications $\xi_1 \to +\infty$, $v_0 = 0$ and $\hat{q}_v \sim \varepsilon$. This means that the displacement of a part F_2 of shell as a rigid body is absent, as well as the corresponding work A_2 and a load parameter are small. It is not contained in the leading asymptotic boundary value problem, which takes the form

$$\frac{d^2\psi}{d\xi^2} + \sin\xi_0 n \frac{d^2\overline{U}}{d\xi^2} = \overline{U}\sin\psi \tag{3.66}$$

$$\frac{d^2\overline{U}}{d\xi^2} - \frac{n}{\sin\xi_0}\frac{d^2\psi}{d\xi^2} = \frac{\cos\psi - \cos\xi_0}{\sin^2\xi_0} \tag{3.67}$$

with boundary conditions

$$\overline{U} = 0, \qquad \psi = \pm\xi_0 \quad \text{as } \xi \to \pm\infty \tag{3.68}$$

Solving this problem numerically for different values of constant n and ξ_0 we can obtain the expression for total potential energy in the following form: $E(\xi_0) = W_1(\xi_0) - A_1(\xi_0)$, and use another variational condition, which was not considered at the first step. Namely, varying $E(\xi_0)$ in ξ_0 we obtain the lacking link between load and parameter of equilibrium configuration ξ_0 in a simple form

$$\frac{\partial W_1}{\partial \xi_0} = \frac{\partial A_1}{\partial \xi_0} \tag{3.69}$$

We have solved problem (3.66)–(3.68) numerically for $n = 0$ and have obtained in accordance with (3.43) and (3.44) the following expression:

$$J = 2.23 + 0.116\xi_0^2 \tag{3.70}$$

which fully evaluates the deformation energy (3.42). Thus, simple explicit formulae (3.42), (3.51), (3.53) together with (3.69) and (3.70) describe the equilibrium path of shell of revolution with large deflections. All shell properties in these formulae are functions $R_1(\xi_0)$, $R_2(\xi_0)$, $\varepsilon(\xi_0)$, $B_{22}(\xi_0)$, $a_*(\xi_0)$, $J(\xi_0)$ that depend on the angle ξ_0 defining the location of the inner boundary layer.

For estimation of the stress state of structure we obtain asymptotic formulae

$$\max|M_1| = \frac{D_{11}\xi_0}{R_1\varepsilon}\max|\varphi'|, \qquad \max|M_2| = \frac{D_{12}}{D_{11}}\max|M_1| \tag{3.71}$$

$$\max|N_2| = a_* B_{22} R_1 \xi_0 \varepsilon \max|f|/R_2, \quad |N_1| \ll |N_2| \tag{3.72}$$

where

$$\max|f| = 0.4 + 0.005\xi_0^2, \qquad \max|\varphi'| = 0.95 + 0.08\xi_0^2 \tag{3.73}$$

In case, when $\xi_0^2 \ll 1$, we obtain results for shallow shells (Evkin, 1989; Evkin and Dubichev, 1997a).

4. Conclusions

The general asymptotic approach for the analysis of anisotropic composite shells of revolution with large axially symmetric deflections is introduced. It allows a significant simplification of the initial governing Reissner's equations. Instead of a very complex boundary value problem with singularities and a large number of stiffness and geometrical parameters of structure, the asymptotic nonlinear boundary-value problem is derived in leading order asymptotic approximation. This boundary value problem is described by the Eqs. (3.29), (3.30), and boundary conditions (3.32)–(3.35). It is free of singularities and it depends only on one shell parameter n defined in the paper. This parameter is equal to zero for a shell symmetrical with respect to its midsurface, or for a shell having eccentricity only in the circumferential direction, when $K_{11} = 0$ and $K_{12} = 0$ in formula (3.18). Further simplification has been achieved by means of variational principle for the complete shells or in the case when inner boundary layer is located sufficiently far from the shell boundary. The explicit asymptotic formulae describing both shell behavior with change of load (3.42), (3.52), (3.53), (3.69), (3.70) and stress states of structure (3.71)–(3.73) have been obtained. It is important to note that the accuracy of obtained asymptotic results coincides with the accuracy of the original boundary value problem for a thin shell which itself is based on the asymptotic limit of three-dimensional equations of theory of elasticity with respect to the same small parameter.

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